Chapter 1  
Basic Probability Review

1.1.

(a) For \((n_1, n_2, n_3)\), let \(n_i\) be the number of demands at \(i\)th distribution center 
\[
\Omega = \{(0,0,0), (1,0,0), (2,0,0), (0,1,0), \cdots, (2,2,2)\}
\]
(b) \(2^3\)
(c) \{ (0,1,2), (0,2,1), (1,1,1), (1,0,2), (1,2,0), (2,0,1), (2,1,0) \}
(d) \(\frac{7}{27}\)

1.3.
\[
\Pr\{N = 0\} = 0.82\\
\Pr\{N = 1\} = 0.15\\
\Pr\{N = 2\} = 0.03
\]

1.5.

(a) \(k = 2\)
(b) 0.6322
(c) 0.8646
(d)
\[
F(t) = \begin{cases} 
0 & \text{for } t < 0.5 \\
1 - e^{-2t} & \text{for } t \geq 0.5 
\end{cases}
\]
(e)
\[
\Pr\{T_1 + T_2 \leq t\} = 1 - e^{-2(t-1)} - 2(t-1)e^{-2(t-1)} \text{ for } t \geq 1 .
\]
Therefore, \(\Pr\{T_1 + T_2 \leq 2\} = 1 - 3e^{-2} = 0.5940\).
(f) Let \(\phi\) be the pdf for \(Y\), then
\[
\phi(t) = \begin{cases} 
0 & \text{for } t < 0.5 \\
2t - 2 + e^{-2t+1} & \text{for } 0.5 \leq t < 1.5 \\
e^{-2t+1}(1 + e^2) & \text{for } t \geq 1.5 .
\end{cases}
\]
1.7. Let \( P \) be the random variable denoting the profit \( E[P] = 510 \)

1.9.

<table>
<thead>
<tr>
<th>Plan</th>
<th>Mean</th>
<th>Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>27500</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>27600</td>
<td>489.9</td>
</tr>
<tr>
<td>C</td>
<td>28200</td>
<td>3340.7</td>
</tr>
</tbody>
</table>

1.11.

\[
E[(X - \mu)^2] = E[X^2 - 2\mu X + \mu^2] \\
= E[X^2] - 2\mu E[X] + \mu^2 \\
= E[X^2] - 2\mu^2 + \mu^2 \\
= E[X^2] - \mu^2
\]

1.13.

\[
E[X] = \int_{x=0}^{b} [1 - F(x)] \, dx = \int_{x=0}^{\infty} [1 - F(x)] \, dx \\
= \int_{x=0}^{\infty} \int_{t=x}^{b} f(t) \, dt \, dx = \int_{t=0}^{\infty} \int_{x=0}^{t} f(t) \, dx \, dt \\
= \int_{t=0}^{\infty} f(t) \left( \int_{x=0}^{t} dx \right) \, dt = \int_{t=0}^{\infty} t f(t) \, dt
\]

1.15.

The number of defective parts in a box follows the Binomial Distribution. Let \( N \) denote the number of defective parts in a box.

(a) \( \Pr\{N = 0\} = 0.8587 \)
(b) \( \Pr\{N = 2\} = 8.214 \times 10^{-3} \)
(c) \( \Pr\{N \geq 2\} = 8.506 \times 10^{-3} \)
(d) \( \Pr\{N \geq 4\} = 0.03377 \)
(e) \( \Pr\{N \geq 20\} = 1 - \Pr\{N < 19.5\} \approx 1 - \Pr\{Z \leq \frac{19.5 - 12}{3.41}\} = 0.0139 \)

1.17.
Let \( \theta \) denote the angle
\( \Pr\{9.9 \leq \theta \leq 10.1\} = \frac{0.2}{0.8} = 0.25 \)

1.19.
\( X \sim N(0, 0.01^2) \), and \( Z \sim N(0, 1) \)
(a) \( \Pr\{-0.005 \leq X \leq 0.005\} = 0.383 \)
(b) \( \Pr\{X \geq 0.02\} + \Pr\{X \leq -0.02\} = 0.0456 \)

1.21.
(a) The marginal pmf for \( X \) is:
\( \Pr\{X = 10\} = 0.1 \)
\( \Pr\{X = 11\} = 0.2 \)
\( \Pr\{X = 12\} = 0.33 \)
\( \Pr\{X = 13\} = 0.4 \)

The marginal pmf for \( Y \) is:
\( \Pr\{Y = 0\} = 0.12 \)
\( \Pr\{Y = 1\} = 0.6 \)
\( \Pr\{Y = 2\} = 0.28 \)

\( E[X] = 12 \)
(b)
\( \Pr\{X = 10 \mid Y = 1\} = 0.1 \)
\( \Pr\{X = 11 \mid Y = 1\} = 0.2 \)
\( \Pr\{X = 12 \mid Y = 1\} = 0.3 \)
\( \Pr\{X = 13 \mid Y = 1\} = 0.4 \)

\( E[X \mid Y = 1] = 12 \)
(c) \( X \) and \( Y \) are not independent.
\( \therefore \Pr\{X = 13, Y = 2\} \neq \Pr\{X = 13\} \times \Pr\{Y = 2\} \)
(d) \( \Pr\{X = 13, Y = 2\} = 0.09 \)
\( \Pr\{X = 13\} = 0.4 \)
\( \Pr\{Y = 2\} = 0.28 \)
\( \Pr\{X = 13, Y = 2\} \neq \Pr\{X = 13\} \times \Pr\{Y = 2\} \)
1.23.  

(a) Let $g_1$ be the marginal pdf for $U$

$$g_1(u) = \int_0^\infty e^{-u-v}dv = e^{-u} \text{ for } u \geq 0$$

Let $g_2$ be the marginal pdf for $V$

$$g_2(v) = \int_0^\infty e^{-u-v}du = e^{-v} \text{ for } v \geq 0$$

$$\Pr\{U \leq 0.5\} = 0.3935$$

$$E[U] = 1$$

(b) $g_{1|v} = e^{-u}$

$$\Pr\{U \leq 0.5 \mid V = 0.1\} = 0.3935$$

$$E[U \mid V = 0.1] = 1$$

(c) $U$ and $V$ are independent

$g(u, v) = g_1(u) \cdot g_2(v)$ for all $u$ and $v$.

1.25.  

$\rho = -0.1429$

1.27.  

It is easier to work with the right-hand-side.


Thus,


1.29.  

First observe that there are $n$ terms in the expression $\sum_{i=1}^{n} X_i^2$ and there are $n(n-1)$ terms in the expression $\sum_{i=1}^{n} \sum_{j\neq i} X_i X_j$. Also due to independence, $E[X_i X_j] = \mu^2$; thus,

$$E[S^2|N = n] = E\left[\left(\sum_{i=1}^{n} X_i\right)^2 \mid N = n\right]$$

$$= E\left[\sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{n} \sum_{j\neq i} X_i X_j\right] = nE[X_i^2] + n(n-1)\mu^2.$$
We let $S$ denote the random sum and we have

$$E[S^2] = E[E[S^2 | N]] = E[N E[X_1^2] + N(N - 1)\mu^2]$$
$$= E[N E[X_1^2]] + \mu^2 E[N^2 - N]$$
$$= E[N] (E[X_1^2] - \mu^2) + \mu^2 E[N^2] = \sigma^2 E[N] + \mu^2 E[N^2],$$

which together with the first moment squared yields the desired result since

$$= \sigma^2 E[N] + \mu^2 E[N^2] - (\mu E[N])^2$$
$$= \sigma^2 E[N] + \mu^2 (E[N^2] - E[N]^2) = \sigma^2 E[N] + \mu^2 V[N].$$