Chapter 1
Basic Probability Review

1.1.

(a) For (n_1, n_2, n_3), let n_i be the number of demands at ith distribution center
$\Omega = \{(0,0,0),\ (1,0,0),\ (2,0,0),\ (0,1,0),\ \cdots,\ (2,2,2)\}$
(b) 2^{27}
(c) { (0,1,2), (0,2,1), (1,1,1), (1,0,2), (1,2,0), (2,0,1), (2,1,0)}
(d) $\frac{7}{27}$

1.3.
$\Pr\{N=0\} = 0.82$
$\Pr\{N=1\} = 0.15$
$\Pr\{N=2\} = 0.03$

1.5.

(a) $k = 2$
(b) 0.6322
(c) 0.8646
(d)

$$F(t) = \begin{cases} 0 & \text{for } t < 0.5 \\ 1 - e^{-2t+1} & \text{for } t \geq 0.5 \end{cases}$$

(e)

$$\Pr\{T_1 + T_2 \leq t\} = 1 - e^{-2(t-1)} - 2(t-1)e^{-2(t-1)} \quad \text{for } t \geq 1\ .$$

Therefore, $\Pr\{T_1 + T_2 \leq 2\} = 1 - 3e^{-2} = 0.5940$.
(f) Let φ be the pdf for Y, then

$$\varphi(t) = \begin{cases} 0 & \text{for } t < 0.5 \\ 2t - 2 + e^{-2t+1} & \text{for } 0.5 \leq t < 1.5 \\ e^{-2t+1}(1+e^2) & \text{for } t \geq 1.5\ . \end{cases}$$

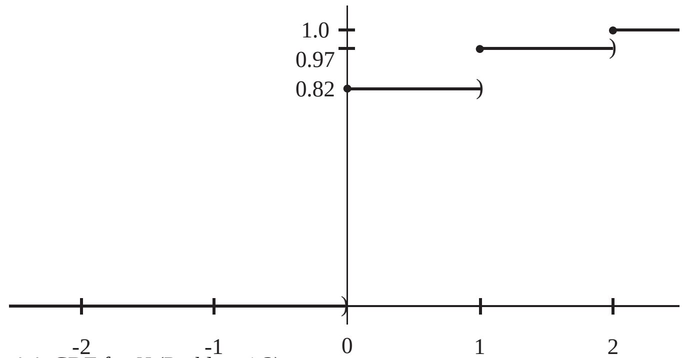

Fig. 1.1 CDF for N (Problem 1.3).

1.7.
Let P be the random variable denoting the profit
$E[P] = 510$

1.9.

Plan	Mean	Std. Dev.
A	27500	0
B	27600	489.9
C	28200	3340.7

1.11.

$$\begin{aligned}E[(X-u)^2] &= E[X^2 - 2\mu X + \mu^2] \\ &= E[X^2] - 2\mu E[X] + \mu^2 \qquad (\because\ E[X] = \mu) \\ &= E[X^2] - 2\mu^2 + \mu^2 \\ &= E[X^2] - \mu^2\end{aligned}$$

1.13.

$$\begin{aligned}E[X] &= \int_0^b [1 - F(x)]\,dx = \int_{x=0}^{\infty} [1 - F(x)]\,dx \\ &= \int_{x=0}^{\infty}\int_{t=x}^{\infty} f(t)\,dt\,dx = \int_{t=0}^{\infty}\int_{x=0}^{t} f(t)\,dx\,dt \\ &= \int_{t=0}^{\infty} f(t)\left(\int_{x=0}^{t} dx\right) dt = \int_{t=0}^{\infty} t\,f(t)\,dt\end{aligned}$$

1.15.
The number of defective parts in a box follows the Binomial Distribution.
Let N denote the number of defective parts in a box.

(a) $\Pr\{N = 0\} = 0.8587$

(b) $\Pr\{N = 2\} = 8.214 \times 10^{-3}$
(c) $\Pr\{N \geq 2\} = 8.506 \times 10^{-3}$
(d)

$$Pr\{N \geq 4\} = 0.03377$$

(e) $\Pr\{N \geq 20\} = 1 - \Pr\{N < 19.5\} \approx 1 - \Pr\{Z \leq \frac{19.5-12}{3.41}\} = 0.0139$

1.17.
Let θ denote the angle
$\Pr\{9.9 \leq \theta \leq 10.1\} = \frac{0.2}{0.8} = 0.25$

1.19.
$X \sim N(0, 0.01^2)$, and $Z \sim N(0,1)$

(a) $\Pr\{-0.005 \leq X \leq 0.005\} = 0.383$
(b) $\Pr\{X \geq 0.02\} + \Pr\{X \leq -0.02\} = 0.0456$

1.21.

(a) The marginal pmf for X is :
$\Pr\{X = 10\} = 0.1$
$\Pr\{X = 11\} = 0.2$
$\Pr\{X = 12\} = 0.33$
$\Pr\{X = 13\} = 0.4$

The marginal pmf for Y is :
$\Pr\{Y = 0\} = 0.12$
$\Pr\{Y = 1\} = 0.6$
$\Pr\{Y = 2\} = 0.28$

$E[X] = 12$
(b)

$\Pr\{X = 10 \mid Y = 1\} = 0.1$

$\Pr\{X = 11 \mid Y = 1\} = 0.2$

$\Pr\{X = 12 \mid Y = 1\} = 0.3$

$\Pr\{X = 13 \mid Y = 1\} = 0.4$

$E[X \mid Y = 1] = 12$
(c) X and Y are not independent.
$\because \Pr\{X = 13, Y = 2\} \neq \Pr\{X = 13\} \times \Pr\{Y = 2\}$
(d) $\Pr\{X = 13, Y = 2\} = 0.09$
$\Pr\{X = 13\} = 0.4$
$\Pr\{Y = 2\} = 0.28$
$\Pr\{X = 13, Y = 2\} \neq \Pr\{X = 13\} \times \Pr\{Y = 2\}$

1.23.

(a) Let g_1 be the marginal pdf for U

$$g_1(u) = \int_0^\infty e^{-u-v} dv = e^{-u} \qquad \text{for} \quad u \geq 0$$

Let g_2 be the marginal pdf for V

$$g_2(v) = \int_0^\infty e^{-u-v} du = e^{-v} \qquad \text{for} \quad v \geq 0$$

$\Pr\{U \leq 0.5\} = 0.3935$

$E[U] = 1$

(b)

$$g_{1|v} = e^{-u}$$

$\Pr\{U \leq 0.5 \mid V = 0.1\} = 0.3935$

$E[U \mid V = 0.1] = 1$

(c) U and V are independent

$\because \quad g(u,v) = g_1(u) \cdot g_2(v)$ for all u and v.

1.25.

$\rho = -0.1429$

1.27.

It is easier to work with the right-hand-side.

$$V[Y|X] = E[Y^2|X] - E[Y|X]^2$$

$$V[\,E[Y|X]\,] = E[\,E[Y|X]^2\,] - E[\,E[Y|X]\,]^2 = E[\,E[Y|X]^2\,] - E[Y]^2$$

Thus,

$$\begin{aligned} E[\,V[Y|X]\,] + V[\,E[Y|X]\,] &= E[\,E[Y^2|X]\,] - E[\,E[Y|X]^2\,] + E[\,E[Y|X]^2\,] - E[Y]^2 \\ &= E[Y^2] - E[Y]^2 = V[Y] \end{aligned}$$

1.29.

First observe that there are n terms in the expression $\sum_{i=1}^n X_i^2$ and there are $n(n-1)$ terms in the expression $\sum_{i=1}^n \sum_{j \neq i} X_i X_j$. Also due to independence, $E[X_i X_j] = \mu^2$; thus,

$$\begin{aligned} E[S^2|N=n] &= E\left[\left(\sum_{i=1}^n X_i \right)^2 |N=n \right] \\ &= E\left[\sum_{i=1}^n X_i^2 + \sum_{i=1}^n \sum_{j \neq i} X_i X_j \right] = nE[X_1^2] + n(n-1)\mu^2 \,. \end{aligned}$$

We let S denote the random sum and we have

$$
\begin{aligned}
E[S^2] &= E[\,E[S^2|N]\,] = E[\,NE[X_1^2] + N(N-1)\mu^2\,] \\
&= E[\,NE[X_1^2]\,] + \mu^2 E[N^2 - N]\,] \\
&= E[N]\left(E[X_1^2] - \mu^2\right) + \mu^2 E[N^2] = \sigma^2 E[N] + \mu^2 E[N^2]\,,
\end{aligned}
$$

which together with the first moment squared yields the desired result since

$$
\begin{aligned}
V[S] &= E[S^2] - E[S]^2 \\
&= \sigma^2 E[N] + \mu^2 E[N^2] - (\mu\, E[N])^2 \\
&= \sigma^2 E[N] + \mu^2\left(E[N^2] - E[N]^2\right) = \sigma^2 E[N] + \mu^2 V[N]\,.
\end{aligned}
$$